

ON APPELGATE–ONISHI'S LEMMAS

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Introduction

In [2] Appelgate and Onishi have proved that the Jacobian Conjecture in the two variables case holds if one of the degrees of the polynomials has at most two prime factors. For the proof they need several key lemmas. Here are two of them (see [2] or Section 1 for notations):

Lemma A. *Let f and g be polynomials in $C[x, y]$ and assume that the jacobian of (f, g) is a non-zero constant. Put $\deg(f) = dm > 1$, $\deg(G) = dn > 1$, where $\gcd(m, n) = 1$. Then for each direction (p, q) , there is a (p, q) -form h of positive degree such that the (p, q) -leading forms of f and g are equal to ah^m and bh^n respectively, for some $a, b \in C^*$.*

Lemma B. *Let f, g, m, n and d be as in Lemma A. Let W_f and W_g be the convex polygons (in the real space \mathbb{R}^2) of f and g , respectively. Then there exists a convex polygon W with vertices in $\mathbb{Z} \times \mathbb{Z}$ such that $W_f = mW$ and $W_g = nW$.*

Lemma A is not proved in [2] and it seems to us that the proof of Lemma B is not clear. But these two lemmas are interesting in themselves and we believe that they will play important roles in the further study of the Jacobian Conjecture. This paper contains our straightforward proofs of them.

1. Preliminaries

Let K be a field of characteristic zero and let $K[x, y]$ be the ring of polynomials in two variables over K .

If $f, g \in K[x, y]$, then we denote by $[f, g]$ the jacobian of (f, g) , that is,

$$[f, g] = f_x g_y - f_y g_x.$$

We shall say that (f, g) is a *basic pair* (or a *basic pair over K*) if $[f, g]$ is a non-zero constant. We denote by K^* the set $K \setminus \{0\}$, and we write $f \sim g$ in the case where $f = ag$, for some $a \in K^*$.

If f is a polynomial in $K[x, y]$, then S_f denotes the *support* of f , that is, S_f is the set of integer points (i, j) such that the monomial $x^i y^j$ appears in f with a non-zero coefficient. W_f is the convex hull (in the real space \mathbb{R}^2) of $S_f \cup \{(0, 0)\}$, and $t_x(f)$ (resp. $t_y(f)$) is the greatest integer s such that the monomial x^s (resp. y^s) appears in f with a non-zero coefficient.

By a *direction* we mean a pair (p, q) of integers such that $\gcd(p, q) = 1$ and $p > 0$ or $q > 0$.

Let (p, q) be a direction. We say that a non-zero polynomial $f \in K[x, y]$ is a (p, q) -form of degree n if f is of the form

$$f = \sum_{pi+qj=n} a_{ij} x^i y^j,$$

where $a_{ij} \in K$, that is, $f \neq 0$ is a (p, q) -form of degree n if and only if the set S_f is contained in the line defined by the equation $px + qy = n$. The degree of a (p, q) -form f is denoted by $d_{pq}(f)$.

Every polynomial $f \in K[x, y]$ has the (p, q) -decomposition $f = \sum_n f_n$ into (p, q) -components f_n of degree n . We denote by f_{pq}^* the (p, q) -component of f of the highest degree. By (p, q) -degree $d_{pq}(f)$ of a polynomial f we mean the number $d_{pq}(f) = d_{pq}(f_{pq}^*)$. In particular we have $d_{11}(f) = \deg(f)$.

Observe that if $pq < 0$, then $d_{pq}(f)$ could be negative. For example, if $p = -2$ and $q = 1$, then $d_{pq}(x^3 y^2 + x^4) = -4$.

The following two well-known facts play a basic role in our considerations:

Lemma 1.1 ([1–3]). *Let f and g be (p, q) -forms of degrees $dm > 0$ and $dn > 0$, respectively, where $\gcd(m, n) = 1$. If $[f, g] = 0$, then there exists a (p, q) -form h of degree d such that $f \sim h^m$ and $g \sim h^n$. \square*

Lemma 1.2 ([1–3]). *Let (f, g) be a basic pair and let (p, q) be a direction. If $d_{pq}(f) + d_{pq}(g) \neq p + q$, then $[f_{pq}^*, g_{pq}^*] = 0$. \square*

The proofs of the next two lemmas are simple.

Lemma 1.3. *If (f, g) is a basic pair, then $(1, 0)$ and $(0, 1) \in S_f \cup S_g$. Moreover, if $(0, 1) \notin S_f$, then $(1, 0) \in S_f$. \square*

Lemma 1.4. Let (f, g) be a basic pair. If f or g belongs to $K[x] \cup K[y]$, then $\min(\deg(f), \deg(g)) = 1$. \square

We end this section with the following:

Proposition 1.5. If (f, g) is a basic pair with $\min(\deg(f), \deg(g)) > 1$, then W_f and W_g are convex polygons, that is, are not line segments.

Proof. Assume that W_f is a line segment. Then, by Lemma 1.3, W_f is a segment of the line $x = 0$ or $y = 0$ (since $(0, 0) \in W_f$), hence $f \in K[x] \cup K[y]$ and, by Lemma 1.4, we have a contradiction. \square

2. The numbers $t_x(f)$ and $t_y(f)$

Let us recall that if $f \in K[x, y]$, then we denote by $t_x(f)$ (resp. $t_y(f)$) the greatest integer s such that the point $(s, 0)$ (resp. $(0, s)$) belongs to the support of f .

Proposition 2.1. If (f, g) is a basic pair with $\min(\deg(f), \deg(g)) > 1$, then the numbers $t_x(f)$, $t_x(g)$, $t_y(f)$ and $t_y(g)$ are all positive.

Proof. Without loss of any generality we may assume that f and g have no constant terms. Suppose that $t_x(f) = 0$ (in the cases $t_y(f) = 0$ or $t_x(g) = 0$ or $t_y(g) = 0$ we do the same procedure). Then f is divisible by y so $f = f_1 y^t$, where $t > 0$ and f_1 is an element of $K[x, y]$ such that $y \nmid f_1$.

Let $t_x(g) = r$. Then, by Lemma 1.3 (since $(1, 0) \notin S_f$), $r > 0$ and we see that $g = w + y g_1$, where $w \in K[x]$, $\deg_x(w) = r$, $g_1 \in K[x, y]$. Therefore we have

$$\begin{aligned} [f, g] &= [f_1 y^t, g] = y^t [f_1, g] + t y^{t-1} f_1 [y, g] \\ &= y^t [f_1, g] - t y^{t-1} f_1 \frac{\partial g}{\partial x} \\ &= y^t [f_1, g] - t y^{t-1} f_1 \frac{\partial w}{\partial x} - t y^{t-1} f_1 y \frac{\partial g_1}{\partial x} \\ &= y^t \left([f_1, g] - t f_1 \frac{\partial g_1}{\partial x} \right) - t y^{t-1} f_1 \frac{\partial w}{\partial x}. \end{aligned}$$

The second term is not divisible by y^t . Hence the assumption $[f, g] \sim 1$ implies, among others, that $t = 1$ and f_1 is a constant. This implies that $\deg(f) = 1$ and we have a contradiction with the assumption that $\min(\deg(f), \deg(g)) > 1$. \square

Corollary 2.2. If (f, g) is a basic pair with $\min(\deg(f), \deg(g)) > 1$, then $d_{pq}(f) > 0$ and $d_{pq}(g) > 0$, for any direction (p, q) .

Proof. Let (p, q) be a direction. Then $p > 0$ or $q > 0$. If $p > 0$, then (by Proposition 2.1)

$$d_{pq}(f) \geq pt_x(f) > 0 \quad \text{and} \quad d_{pq}(g) \geq pt_x(g) > 0.$$

If $q > 0$, then (by Proposition 2.1)

$$d_{pq}(f) \geq qt_y(f) > 0 \quad \text{and} \quad d_{pq}(g) \geq qt_y(g) > 0. \quad \square$$

Corollary 2.3. *If (f, g) is a basic pair with $\min(\deg(f), \deg(g)) > 1$, then $d_{pq}(f) + d_{pq}(g) > p + q$, for any direction (p, q) .*

Proof. If $p > q$, then $p > 0$ and, by Proposition 2.1, $d_{pq}(f) + d_{pq}(g) \geq pt_x(f) + pt_x(g) \geq p + p > p + q$. In a similar way we test the case when $p < q$.

If $p = q$, then $p = q = 1$ (since $\gcd(p, q) = 1$) and we have $d_{pq}(f) + d_{pq}(g) = \deg(f) + \deg(g) > 2 = p + q$. \square

3. The similarity of polygons W_f and W_g

If $h \in K[x, y]$, then we denote by W_h^* the set of non-zero vertices of W_h . Denote also by 0 the point $(0, 0)$.

Lemma 3.1. *Let (f, g) be a basic pair with $\min(\deg(f), \deg(g)) > 1$. Then for every vertex $A \in W_f^*$ there exists the unique vertex $B \in W_g^*$ such that the points A, B and 0 are collinear.*

Proof. Let $A = (a, b) \in W_f^*$. Let (p, q) be a direction such that A is the (p, q) -leading point in S_f . Then $f_{pq}^* \sim x^a y^b$ and, by Corollaries 2.2 and 2.3, $d_{pq}(f) > 0$, $d_{pq}(g) > 0$ and $d_{pq}(f) + d_{pq}(g) > p + q$. Hence, by Lemmas 1.1 and 1.2, there exists a (p, q) -form h (of positive degree) such that $f_{pq}^* \sim h^s$ and $g_{pq}^* \sim h^t$, for some $s \geq 1$ and $t \geq 1$. But $f_{pq}^* \sim x^a y^b$, so h is a monomial in $K[x, y]$. Let $h \sim x^i y^j$, for some $i \geq 0$, $j \geq 0$ such that $i + j > 0$. Then $(a, b) = (si, sj)$ and, since $g_{pq}^* \sim x^{it} y^{jt}$, we see that $B = (it, jt)$ is a non-zero vertex of W_g . The points $A, B, 0$ lie on the line $jx - iy = 0$ and it is clear that B is unique. \square

Lemma 3.2. *Let (f, g) be a basic pair with $\min(\deg(f), \deg(g)) > 1$. Let $A \neq A'$ be non-zero vertices of W_f such that the segment AA' is an edge of W_f . Let B be the unique vertex of W_g such that $0, A, B$ are collinear (see Lemma 3.1) and let B' be the unique vertex of W_g such that $0, A', B'$ are collinear. Then $B \neq B'$ and the segment BB' is an edge of W_g , and the segments AA' and BB' are parallel.*

Proof. It is clear that $B \neq B'$. Moreover, if the segment BB' is not an edge of W_g , then in the angle $(0B, 0B')$ lies a non-zero vertex of W_g different from B and B' .

and then, by Lemma 3.1, there exists a vertex of W_f belonging to the same angle and different from A and A' ; it is a contradiction with the assumption that AA' is an edge of W_f . Therefore BB' is an edge of W_g .

Let (p, q) be the direction of the line AA' . By Corollaries 2.2, 2.3 and Lemmas 1.1 and 1.2, there exists a (p, q) -form h such that $f_{pq}^* \sim h^s$ and $g_{pq}^* \sim h^t$, for some $s > 0$ and $t > 0$.

Let $d = d_{pq}(h)$.

Since A and A' belong to the support of f_{pq}^* , we see that h is not a monomial. Assume that

$$h = u_1 x^{a_1} y^{b_1} + \cdots + u_r x^{a_r} y^{b_r},$$

where u_1, \dots, u_r are non-zero constants, $r \geq 2$, and $(a_i, b_i) \neq (a_j, b_j)$ for any $i \neq j$, and $pa_i + qb_i = d$ for any $i = 1, \dots, r$.

Case I. Let $p = 0$. Then $q = 1$, $b_1 = \cdots = b_r = d$ and we may assume that $a_1 < \cdots < a_r$. It is easy to see that then $A = (a_1 s, ds)$, $A' = (a_r s, ds)$, $B = (a_1 t, dt)$ and $B' = (a_r t, dt)$. Hence the segments AA' and BB' are parallel.

If $q = 0$, then we apply the same procedure.

Case II. Let $p \neq 0$ and $q \neq 0$. Then $a_i \neq a_j$ and $b_i \neq b_j$ if $i \neq j$, so we may assume that $a_1 < \cdots < a_r$ or $b_1 < \cdots < b_r$, and then we have $A = (a_1 s, b_1 s)$, $A' = (a_r s, b_r s)$, $B = (a_1 t, b_1 t)$ and $B' = (a_r t, b_r t)$. Therefore AA' and BB' are parallel. \square

As an immediate consequence of Lemmas 3.1 and 3.2 we obtain the following:

Corollary 3.3. *If (f, g) is a basic pair with $\min(\deg(f), \deg(g)) > 1$, then the polygons W_f and W_g are similar.* \square

4. Proofs of Lemmas A and B.

Using our terminology Lemmas A and B (see the introduction) will be reformulated in the following form:

Lemma A. *Let K be a field of characteristic zero and let (f, g) be a basic pair over K , such that $\deg(f) = dm > 1$, $\deg(g) = dn > 1$, where $\gcd(m, n) = 1$. Then, for each direction (p, q) , there exists a (p, q) -form h of positive degree such that $f_{pq}^* \sim h^m$ and $g_{pq}^* \sim h^n$.*

Lemma B. *Let K, f, g, m, n and d be as in Lemma A. Then there exists a convex polygon W with vertices in $\mathbb{Z} \times \mathbb{Z}$ such that $W_f = mW$ and $W_g = nW$.*

Proof of Lemma B. We know, by Section 3, that W_f and W_g are similar and we know that if A_1, \dots, A_s are successive non-zero vertices of W_f , then we have the

sequence B_1, \dots, B_s of successive non-zero vertices of W_g such that the points $0, A_i, B_i$ are collinear for $i = 1, \dots, s$. Therefore the numbers $|0A_i|/|0B_i|$ (we denote by $|UV|$ the length of a segment UV) are the same, say are equal to e , for any $i = 1, \dots, s$.

Now consider the direction $(1, 1)$. Since $d_{11}(f) = \deg(f) = dm > 1$ and $d_{11}(g) = \deg(g) = dn > 1$, there exists a $(1, 1)$ -form h of degree d such that $f_{11}^* \sim h^m$ and $g_{11}^* \sim h^n$ (by Lemmas 1.1 and 1.2).

Let $h = u_1 x^{a_1} y^{b_1} + \dots + u_r x^{a_r} y^{b_r}$, where $r \geq 1, u_1, \dots, u_r$ are non-zero constants and $a_1 < \dots < a_r$. Then the point $A = (ma_1, mb_1)$ belongs to W_f^* , and the point $B = (na_1, nb_1)$ belongs to W_g^* and the points $A, B, 0$ are collinear. Therefore $e = |0A|/|0B| = m/n$.

Fix $i \in \{1, \dots, s\}$. Let $A_i = (x_i, y_i)$ and $B_i = (z_i, t_i)$. Then $(nx_i, ny_i) = (mz_i, mt_i)$ and, since $\gcd(m, n) = 1$, there exists $(w_i, v_i) \in \mathbb{Z} \times \mathbb{Z}$ such that $A_i = (mw_i, mv_i)$ and $B_i = (nw_i, nv_i)$.

Let W be the polygon with vertices $(0, 0), (w_1, v_1), \dots, (w_s, v_s)$. Then W is convex and we see that $W_f = mW, W_g = nW$. This completes the proof of Lemma B. \square

Proof of Lemma A. Let (p, q) be a direction. Since (by Corollaries 2.2 and 2.3) $d_{pq}(f) > 0$, and $d_{pq}(g) > 0$ and $d_{pq}(f) + d_{pq}(g) > p + q$, there exists a (p, q) -form h such that $f_{pq}^* \sim h^{m'}$ and $g_{pq}^* \sim h^{n'}$, for some natural m', n' with $\gcd(m', n') = 1$ (by Lemmas 1.1 and 1.2). Now it is clear that $d_{pq}(h) > 0$.

We shall show that $m = m'$ and $n = n'$. Consider a vertex (a, b) of W_h . Then $(m'a, m'b), (n'a, n'b)$ are non-zero vertices of W_f and W_g , respectively. Therefore, by Lemma B, $m'/n' = m/n$, that is, $m' = m$ and $n' = n$ (since $\gcd(m', n') = \gcd(m, n) = 1$). This completes the proof of Lemma A. \square

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